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ELASTIC INTERACTION OF A CRACK WITH A MICROCRACK ARRAY—II. ELASTIC SOLUTION FOR TWO CRACK CONFIGURATIONS (PIECEWISE CONSTANT AND LINEAR APPROXIMATIONS)

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Abstract—The approach to crack–microcrack array interaction problems developed in Part I is applied here to two configurations involving one and two microcracks. These configurations, although simple, exhibit the important effects of stress 'shielding' and stress 'amplification' (reduction or increase of the effective stress intensity factor due to the presence of microcracks) occurring in more complex microcrack arrays; they also illustrate the refinements introduced by higher order polynomial approximations.

1. MICROCRACK COLLINEAR TO THE MAIN CRACK (PIECEWISE CONSTANT APPROXIMATION)

The problem of elastic interaction of the main crack $(-l_0, l_0)$ with one microcrack $(c - l, c + l)$ located on a continuation of the main crack's line (Fig. 1) is considered. Plane stress conditions and Mode I uniform remote loading are assumed; the microcrack is assumed to be embedded into the main crack tip stress field ('small scale microcracking'). In a piecewise constant approximation, we approximate this field within the microcrack line $(c - l, c + l)$ by a constant equal to the value of the field at the microcrack center $x = c$. Because of symmetry, $\sigma_{yy} = \sigma_{yy}(c)$ is the only stress component acting along the microcrack line. Thus, the overall stress field in the vicinity of the main crack tip is given by a superposition

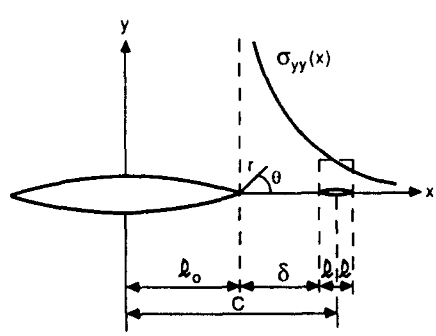

Fig. 1.

$$\sigma(\mathbf{x}) = \hat{\sigma}(\mathbf{x}) + \sigma^l(\mathbf{x})$$

where

$$\hat{\sigma}(\mathbf{x}) = K_{\mathrm{I}}^{\mathrm{eff}} \frac{\varphi[\theta(\mathbf{x})]}{\sqrt{(2\pi r(\mathbf{x}))}} \tag{1}$$

$$\sigma^l(\mathbf{x}) = \mathbf{T}_x \int_{c-l}^{c+l} \mathbf{b}(\xi)\cdot\mathbf{\Phi}(\xi,\mathbf{x})\,\mathrm{d}\xi \tag{2}$$

and $\mathbf{T}_x\{\mathbf{u}(\mathbf{x})\}$ denotes the stress operator transforming a displacement field into stresses in accordance with Hooke's law: $T_{ij}\{u\} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij}$ (index "x" indicates that the differentiation is performed with respect to $\mathbf{x}$). The microcrack's COD is elliptical (since in a piecewise constant approximation the microcrack is subjected to a uniform stress $\sigma_{yy}(c)$) and is given by

$$\mathbf{b}(\xi) = \frac{4l}{E}\frac{K_{\mathrm{I}}^{\mathrm{eff}}}{\sqrt{(2\pi(l+\delta))}}e(\xi)\mathbf{n} \tag{3}$$

where E is Young's modulus, $\mathbf{n}$ the unit vector in the y-direction and $e(\xi) = \sqrt{(1-(\xi-c)^2/l^2)}$ is an ellipse of the unit opening with the extremities at the microcrack tips. Substitution of eqn (3) into eqn (2) and introduction of the stress operator $\mathbf{T}_x$ under the integral yields

$$\sigma_{ij}^l(\mathbf{x}) = \frac{4l}{E}\frac{K_{\mathrm{I}}^{\mathrm{eff}}}{\sqrt{(2\pi(l+\delta))}}\int_{c-l}^{c+l}\left\{\mu[\Phi_{2i,j}(\xi,\mathbf{x}) + \Phi_{2j,i}(\xi,\mathbf{x})] + \lambda\Phi_{2k,k}(\xi,\mathbf{x})\delta_{ij}\right\}e(\xi)\,\mathrm{d}\xi \tag{4}$$

After integration $\sigma_{yy}^l(x)$ appears in the form

$$\sigma_{yy}^l(x) = \frac{K_{\mathrm{I}}^{\mathrm{eff}}}{\sqrt{(2\pi(l+\delta))}}\left[\frac{1}{\sqrt{(1-l^2/(x-c)^2)}} - 1\right]. \tag{5}$$

Substitution into the equation for effective stress intensity factor

$$K_{\mathrm{I}}^{\mathrm{eff}} = K_{\mathrm{I}}^{0} + \frac{1}{\sqrt{(\pi l_0)}}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+\xi}{l_0-\xi}\right)}\sigma_{yy}^l(\xi)\,\mathrm{d}\xi \tag{6}$$

($K_{\mathrm{I}}^0 = \sigma_\infty\sqrt{(\pi l_0)}$ is the stress intensity factor in the absence of the microcrack) yields the following equation for $K_{\mathrm{I}}^{\mathrm{eff}}$: $K_{\mathrm{I}}^{\mathrm{eff}} = K_{\mathrm{I}}^0 + qK_{\mathrm{I}}^{\mathrm{eff}}$, so that

$$K_{\mathrm{I}}^{\mathrm{eff}} = \frac{K_{\mathrm{I}}^0}{1-q} \tag{7}$$

where

$$q = \frac{1}{\sqrt{(2(l'+\delta'))}}\int_{-1}^{1}\sqrt{\left(\frac{1+t}{1-t}\right)}\frac{1}{\sqrt{(1-l'^2/(t-c')^2)}} - 1\,\mathrm{d}t \tag{8}$$

and $l' = l/l_0$, $\delta' = \delta/l_0$, $c' = c/l_0$ are nondimensional geometrical parameters. The graph of $K_{\mathrm{I}}^{\mathrm{eff}}/K_{\mathrm{I}}^0$ vs δ/l is given in Figs 2 and 3. When the distance between two cracks tends to

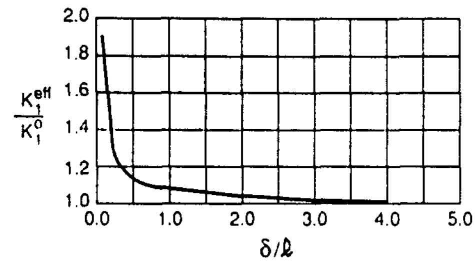

Fig. 2.

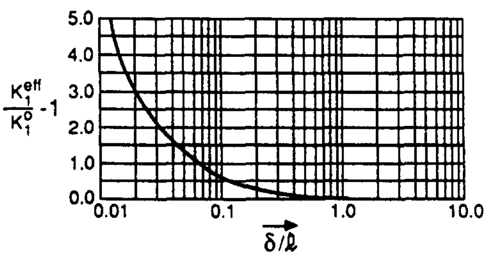

Fig. 3.

zero $K_{\rm I}^{\rm eff} \to \infty$.

The resulting stress field in the vicinity of the main crack tip can now be written as

$$\sigma_{ij}(\mathbf{x}) = \hat{\sigma}_{ij}(\mathbf{x}) + \sigma_{ij}^{l}(\mathbf{x}) =$$

$$K_{\rm I}^{\rm eff}\left\{\frac{\varphi_{ij}[\theta(\mathbf{x})]}{\sqrt{(2\pi r(\mathbf{x}))}} + \frac{4l}{E}\frac{1}{\sqrt{(2\pi(l+\delta))}}\int_{c-l}^{c+l}\mu(\Phi_{2i,j}+\Phi_{2j,i}) + \lambda\Phi_{2k,k}\delta_{ij}e(\xi)\,\mathrm{d}\xi\right\} \tag{9}$$

where $K_{\rm I}^{\rm eff}$ is given by eqn (7). This expression gives the solution of the problem in the piecewise constant approximation.

The problem considered involves just one microcrack and the iterative approach is unnecessary; furthermore, $K_{\rm I}^{\rm eff}$ is obtained in the form of eqn (7) even if polynomial approximations of higher orders are used (see Section 2). It appears to be of interest, however, to find physical meaning of the iterations. Solving the equation $K_{\rm I}^{\rm eff} = K_{\rm I}^{0} + qK_{\rm I}^{\rm eff}$ by iterations yields: $K_{\rm I}^{\rm eff} = K_{\rm I}^{0}$; $K_{\rm I}^{\rm eff} = (1+q)K_{\rm I}^{0}$; $K_{\rm I}^{\rm eff} = (1+q+q^2)K_{\rm I}^{0}$; ...—a geometrical series representation of the solution $K_{\rm I}^{\rm eff} = K_{\rm I}^{0}(1-q)$. Having in mind that the first two terms in braces in eqn (9) give the main crack tip field and the microcrack-generated field, correspondingly, and that the parameter q reflects the impact of the microcrack-generated field on $K_{\rm I}^{\rm eff}$, the iterations can be interpreted by means of the diagram in Fig. 4. The first line presents terms in $\sigma(\mathbf{x})$ corresponding to $K_{\rm I}^{\rm eff} = K_{\rm I}^{0}$ (zeroth iteration), the second line corresponds to the term $qK_{\rm I}^{0}$ added to $K_{\rm I}^{\rm eff}$ in the first iteration and the third line corresponds to the term $q^2K_{\rm I}^{0}$ added in the second iteration. Thus, the higher order iterations represent the crack interactions of higher multiplicity.

2. MICROCRACK COLLINEAR TO THE MAIN CRACK (HIGHER ORDER APPROXIMATIONS)

The same problem is reconsidered here on the basis of higher order polynomial approximations, and the corrections to the piecewise constant approximation are examined.

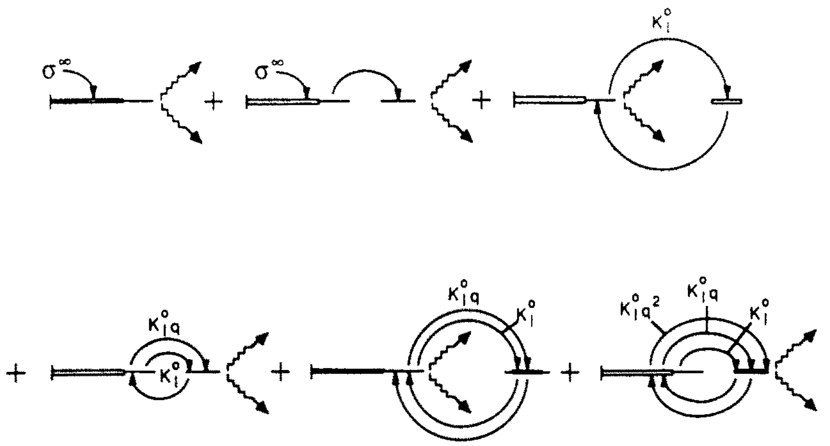

Fig. 4.

2.1. *Linear approximation*

In the linear approximation, the stress $\hat{\sigma}_{yy}$ induced along the microcrack line l by the main crack tip is assumed to change linearly along l (since $\hat{\sigma}_{yy}$ is the only relevant stress component — other components vanish along the line of the crack, the yy-index is omitted in this section)

$$\hat{\sigma}(x) = \hat{\sigma}(c) + \hat{\sigma}'(c)(x - c) \tag{10}$$

where

$$\hat{\sigma}(c) = K_{\mathrm{I}}^{\mathrm{eff}}/\sqrt{(2\pi(l + \delta))} \tag{11}$$

$$\hat{\sigma}'(c) = \left(\frac{\partial \hat{\sigma}}{\partial x}\right)_{x=c} = -K_{\mathrm{I}}^{\mathrm{eff}}/2(l + \delta)\sqrt{(2\pi(l + \delta))} \tag{12}$$

so that, in dimensionless form

$$\hat{\sigma}(x') = \frac{K_{\mathrm{I}}^{\mathrm{eff}}}{\sqrt{l_0}} \frac{1}{\sqrt{(2\pi(l' + \delta'))}} \left[1 - \frac{x' - c'}{2(l' + \delta')}\right]. \tag{13}$$

According to the polynomial conservation theorem, a polynomial loading of degree P generates the elliptical COD multiplied by the polynomial of the same degree P. Therefore, instead of eqn (10), we have

$$b(\xi) = [b_0 + b_1(\xi' - c')]4e(\xi) \tag{14}$$

where b_0 and b_1 are certain coefficients of the length dimensions. According to eqn (2), the COD, eqn (3), corresponds to the following traction on the microcrack line

$$\sigma^l(x') = \frac{E}{\pi l} \int_{c-l}^{c+l} \frac{e(\xi)}{(\xi' - x')^2} [b_0 + b_1(\xi' - c')] \, \mathrm{d}\xi. \tag{15}$$

Evaluation of the integral when $x' \in (c' - l', c' + l')$ yields

$$\sigma^l(x') = \frac{E}{l} [b_0 + 2b_1(x' - c')]. \tag{16}$$

(Evaluating the integral we used the regularization described in Part I.) Comparing two different expressions, eqns (10) and (16), for the same linear function, one obtains: $b_0 = (l/E)\hat{\sigma}(c')$, $b_1 = (1/2)(l/E)\hat{\sigma}'(c')$. Substitution of the later relations into eqn (15) results in the expression for the stress σ^l generated by the microcrack on the continuation of its line (i.e. along the x-axis outside the interval $(c - l, c + l)$) in the form of linear combination of $\hat{\sigma}(c')$ and $\hat{\sigma}'(c')$

$$\sigma^l(x') = \hat{\sigma}(c')\left\{\left[1 - \left(\frac{l'}{x' - c'}\right)^2\right]^{-1/2} - 1\right\} - \frac{1}{2}(c' - x')\hat{\sigma}'(c')\left\{\left[1 - \left(\frac{l'}{x' - c'}\right)^2\right]^{-1/2} + \left[1 - \left(\frac{l'}{x' - c'}\right)^2\right]^{1/2} - 2\right\}. \tag{17}$$

Substituting eqn (17) into expression (6) for K_I^{eff} results in the equation which, together with eqns (11) and (12), constitutes a system of three linear algebraic equations for $\sigma(c)$, $\sigma'(c)$ and K_I^{eff}. The solution for K_I^{eff} yields eqn (7) with

$$q = \frac{1}{\pi\sqrt{(2(l' + \delta'))}} f_0(l, \delta) + \frac{1}{4\pi(l' + \delta')\sqrt{(2(l' + \delta'))}} f_1(l, \delta) \equiv q_0 + q_1 \tag{18}$$

where the following notations are introduced

$$f_0(l, \delta) = \int_{-1}^{1} \sqrt{\left(\frac{1 + x}{1 - x}\right)} I_0(c', x')\,\mathrm{d}x \tag{19}$$

$$f_1(l, \delta) = \int_{-1}^{1} \sqrt{\left(\frac{1 + x}{1 - x}\right)} [I_1(c', x') + I_0(c', x')(c' - x')]\,\mathrm{d}x \tag{20}$$

$$I_0(c', x') = l \int_{c-l}^{c+l} \frac{e(\xi)}{(\xi - x)^2}\,\mathrm{d}\xi = \pi\left\{\left[1 - \left(\frac{l}{x - c}\right)^2\right]^{-1/2} - 1\right\} \tag{21}$$

$$I_1(c', x') = l \int_{c-l}^{c+l} \frac{e(\xi)}{\xi - x}\,\mathrm{d}\xi = \pi(x' - c')\left\{\left[1 - \left(\frac{l}{x - c}\right)^2\right]^{1/2} - 1\right\}. \tag{22}$$

The coefficients q_0 and q_1, are responsible for an increase of the effective stress intensity factor K_I^{eff} due to elliptic microcrack opening and due to linear deviation of elliptic shape, correspondingly.

General superposition formula gives the resulting stress field in the form

$$\sigma(\mathbf{x}) = K_I^{\text{eff}}\left\{\frac{\varphi[\theta(\mathbf{x})]}{\sqrt{(2\pi r(\mathbf{x}))}} + \frac{4l}{E}\cdot\frac{1}{\sqrt{(2\pi(l + \delta))}}\mathbf{T}_x \int_{c-l}^{c+l}\left[1 - \frac{\xi - c}{4(l + \delta)}\right] e(\xi)\mathbf{n}(c)\cdot\mathbf{\Phi}(\xi, \mathbf{x})\,\mathrm{d}\xi\right\}. \tag{23}$$

The second term in brackets in the integrand represents the correction due to the linear approximation (as compared to the piecewise constant approximation). Figure 5 shows the graphs of K_I^{eff}/K_I^0 for the linear (upper curve) and piecewise constant (lower curve) approximations. The curves diverge when the distance between the cracks becomes very small.

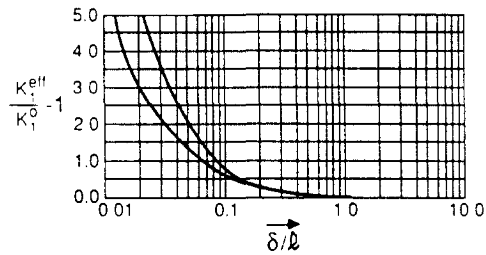

Fig. 5.

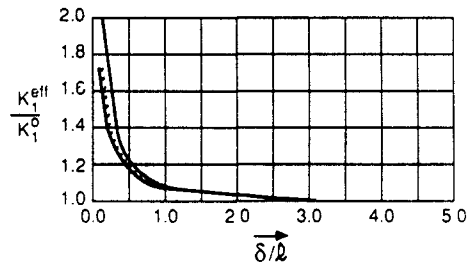

Fig. 6.

It should be noted that the solution obtained gives a low estimate for both k_1^{eff} and the stress field $\sigma(\mathbf{x})$, since it corresponds to an approximation of the curve $\hat{\sigma}(x)$ by the straight line tangent to the curve at the center C of the microcrack. The upper bound can be obtained by taking a chord passing through the end points of the microcrack. The result of calculations based on the chord linear approximation are shown in Fig. 6 (upper curve), together with the results based on the tangent linear approximation (lower curve). The dashed curve represents the *exact* solution for K_1^{eff} taken from Ref. [1] and is expressible in elliptic functions. The two linear approximations yield results which are close to each other and to the exact solution unless the distance between the cracks becomes very small.

2.2. *Higher order approximations*

Although, as Fig. 6 shows, linear approximation gives good agreement with the exact solution in the problem considered, the structure of the higher order approximations appear to be of interest.

In the quadratic approximation, the main crack tip field $\hat{\sigma}$ is approximated by a second-order Taylor's polynomial on the microcrack

$$\hat{\sigma}(x') = \hat{\sigma}(c') + \hat{\sigma}'(c')(x' - c') + \hat{\sigma}''(c')\frac{(x' - c')^2}{2!} \tag{24}$$

where

$$\hat{\sigma}^{(p)}(c') = K_1^{eff}\frac{d^p}{dx^p}\left(\frac{1}{\sqrt{(2\pi x')}}\right)_{x'=c'}; \qquad p = 0, 1, 2. \tag{25}$$

This, according to the polynomial conservation theorem, implies that the microcrack's COD can be represented in the form

$$b(x') = \left[b_0 + b_1(x' - c') + b_2 \frac{(x' - c')^2}{2!} \right] 4e(x') \tag{26}$$

which, in accordance with the potential representation, eqn (2), corresponds to the following traction on the microcrack

$$\sigma^l(x') = (\mathbf{n}\,\mathbf{n}) : \mathbf{T}_{x'} \int_{c'-l'}^{c'+l'} \sum_{p=0}^{2} b_p \frac{(\xi' - c')^p}{p!} e(\xi')\mathbf{n} \cdot \mathbf{\Phi}(\xi', x')\,\mathrm{d}\xi'. \tag{27}$$

This traction is a second degree polynomial, and, by comparing the coefficients of eqns (27) and (24), a linear relationship between the derivatives $\hat{\sigma}^{(p)}(c)$ of the main crack tip field and the coefficients b_p can be established

$$\hat{\sigma}^{(p)(c)} = \sum_{k=0}^{2} B_{pk}^{-1}\, b_k \tag{28}$$

or, in matrix form

$$\{\hat{\sigma}^{(p)}\} = \{B\}^{-1}\{b\}. \tag{29}$$

Evaluation of the integrals in eqn (27) yields the following transformation matrix $\{B\}^{-1}$

$$\{B\}^{-1} = \frac{E}{l} \begin{pmatrix} 1 & 0 & \frac{1}{4} l'^2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \tag{30}$$

(for a linear approximation, it degenerates into a 2×2 matrix obtained from eqn (30) by crossing out the third row and the third column).

To find the microcrack-generated stress (in terms of its COD) and the corresponding corrections to $K_{\mathrm{I}}^{\mathrm{eff}}$ system of eqns (29) has to be solved. The inverse matrix is

$$\{B\} = \frac{l}{E} \begin{pmatrix} 1 & 0 & -\frac{l'^2}{3 \cdot 4} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \tag{31}$$

and $\{b\} = \{B\}\{\hat{\sigma}^{(p)}\}$. Substituting $\{\hat{\sigma}^{(p)}\}$ from eqn (25) in the latter and substituting the potential representation for the microcrack-generated stress $\sigma^l(x')$ into eqn (6) for $K_{\mathrm{I}}^{\mathrm{eff}}$ one obtains a linear algebraic equation for $K_{\mathrm{I}}^{\mathrm{eff}}$

$$K_{\mathrm{I}}^{\mathrm{eff}} = K_{\mathrm{I}}^{0} + (q_0 + q_1 + q_2)K_{\mathrm{I}}^{\mathrm{eff}} \tag{32}$$

with the terms in parentheses corresponding to piecewise constant, linear and quadratic approximations, respectively. The quantities q_0, q_1, and q_2 are given by eqns (8) and (18), and

$$q_2 = \frac{1}{\pi}\sqrt{\left(\frac{l_0}{\pi}\right)}\frac{1}{2!3}\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{(2\pi x)}}\right)_{x=c}\int_{-1}^{1}\sqrt{\left(\frac{1+x'}{1-x'}\right)}[(x'-c')^2 I_0 + 2(x'-c')I_1 + I_2]\,dx' \tag{33}$$

where I_0 and I_1 are given by eqns (21) and (22) and

$$I_2 = -\int_{c'-l'}^{c'+l'} e(t)\,dt = -\frac{\pi l^2}{2}. \tag{34}$$

Note that, as seen from the structure of the matrix $\{B\}$, the even (or odd) order coefficients b_0, b_2 (or b_1) are related to the even (odd) order derivatives $\hat{\sigma}^{(p)}$ only. Note, also, that the integrals I_0, I_1 and I_2 can be expressed in terms of elementary functions. These observations hold, as shown below, for higher order approximations as well.

In the general case of the Pth order approximation, the main crack tip stress field along the microcrack line is represented as

$$\sigma(x) = \sum_{k=0}^{P} \sigma_{yy}^{(k)}(c)\frac{(x-c)^k}{k!} \tag{35}$$

so that the microcrack's COD is given by

$$b(x) = 4e(x)\sum_{k=0}^{P} b_x \frac{(x-c)^k}{k!}. \tag{36}$$

Proceeding as above one arrives at the same expression, eqn (7) for K_I^{eff}, with

$$q = \sum_{n=0}^{P} q_n \tag{37}$$

where

$$q_n = \frac{1}{\pi\sqrt{(\pi l_0)}}\sum_{k=0}^{P} B_n \frac{d^k}{dx^k}\left(\frac{1}{\sqrt{(2\pi x)}}\right)_{x=c}\int_{-l_0}^{l_0}\sqrt{\left(\frac{l_0+x}{l_0-x}\right)}\left\{\int_{c-l}^{c+l}\frac{(\xi-c)^n}{n!}\frac{e(\xi)}{(\xi-x)^2}d\xi\right\}dx. \tag{38}$$

Matrix $\{B\}$ relating the derivatives of the main crack tip field and the polynomial coefficients of the microcrack's COD has the structure similar to the one in the quadratic approximation

$$\{B\} = \frac{l}{E}\begin{pmatrix} 1 & 0 & \beta_{02}l'^2 & 0 & \beta_{04}l'^4 & 0 & \\ 0 & \frac{1}{2} & 0 & \beta_{13}l'^2 & 0 & \beta_{15}l'^4 & \\ 0 & 0 & \frac{1}{3} & 0 & \beta_{24}l'^2 & 0 & \\ - & - & - & - & - & - & \\ 0 & 0 & 0 & - & - & - & \frac{1}{P} \end{pmatrix} \tag{39}$$

where coefficients β_{ij} are linear combinations of integrals of the type $R(x, \sqrt{(x^2+ax+b)})$ (R denotes a rational function) and can, therefore, be expressed in terms of elementary functions (i, j are not tensorial indices here).

As $P \to \infty$, formulas (35) and (36) become the Taylor's series for the analytic fields

$\hat{\sigma}(x)$ and $b(x)$. The rate of their convergence to the exact solution will depend on the ratio δ/l. Indeed, the convergence in formula (38) depends on the behavior of the following series

$$\sum_{n\geq k}\sum_{k>0} B_{nk}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{1}{\sqrt{(2\pi x)}}\right)_{x=c}\int_{c-l}^{c+l}\frac{(\xi-c)^n}{n!}f(\xi,x)\,\mathrm{d}\xi \tag{40}$$

where

$$f(\xi,x)=e(\xi)/(\xi-x)^2\cdot\sqrt{\left(\frac{l_0+x}{l_0-x}\right)}.$$

Taking into account that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{1}{\sqrt{(2\pi x)}}\right)_{x=c}=\frac{(-1)^k(2k-1)!!}{\sqrt{(2\pi(l+\delta))}(l+\delta)^k2^k} \tag{41}$$

and that

$$\left|\int_{c-l}^{c+l}\frac{(\xi-c)^n}{n!}f(\xi,x)\,\mathrm{d}\xi\right|\leqslant\frac{l^n}{n!}\left|\int_{c-l}^{c+l}f(\xi,x)\,\mathrm{d}\xi\right| \tag{42}$$

can be bounded as follows

$$\begin{aligned}&\left|\sum_{n>k>0}\sum B_{nk}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{1}{\sqrt{(2\pi x)}}\right)_{x=c}\int_{c-l}^{c+l}\frac{(\xi-c)^n}{n!}f(\xi,x)\right|\\&\qquad\leqslant\left|\frac{1}{\sqrt{(2\pi(l+\delta))}}\sum_{n>k>0}\sum\beta_{nk}\frac{(2k-1)!!}{2^kn!}\cdot\frac{1}{(1+\delta/l)^k}\int_{c-l}^{c+l}f(\xi,x)\,\mathrm{d}\xi\right|\end{aligned} \tag{43}$$

one observes that the series on the right-hand side of inequality (43) is absolutely convergent and the rate of convergence depends on δ/l.

3. TWO MICROCRACKS PARALLEL TO THE MAIN CRACK

The configuration considered is shown in Fig. 7. Plane stress conditions, mode I uniform remote loading, and 'small scale microcracking' are assumed. The problem will be considered in the piecewise constant approximation. Two different effects of crack interaction may occur depending on the relative values of the geometrical parameters; one when $c \gg l_0$ and the microcracks 'amplify' the stress concentration ($K_\mathrm{I}^{\mathrm{eff}} > K_\mathrm{I}^0$), the other when $c \simeq l_0$ and the microcracks 'shield' the macrocrack tip ($K_\mathrm{I}^{\mathrm{eff}} < K_\mathrm{I}^0$). In the intermediate range of c/l_0 these two effects compete.

Because of symmetry the boundary conditions on both microcracks are the same. Therefore, the system of eqns (7) of Part I can be rewritten for this case as follows

$$\mathbf{n}\cdot\mathbf{T}_x\left\{\int_{l_1}\mathbf{b}(x')\cdot\boldsymbol{\phi}(x',\mathbf{x})\,\mathrm{d}x'+\int_{l_2}\mathbf{b}(x')\cdot\boldsymbol{\phi}(x',\mathbf{x})\,\mathrm{d}x'\right\}=-K_\mathrm{I}^{\mathrm{eff}}\,\mathbf{n}\cdot\boldsymbol{\sigma}_0(\mathbf{x}) \tag{44}$$

where $x \in l_1$ and the CODs of both microcracks are denoted by $\mathbf{b}$. Under the assumption of piecewise constant approximation

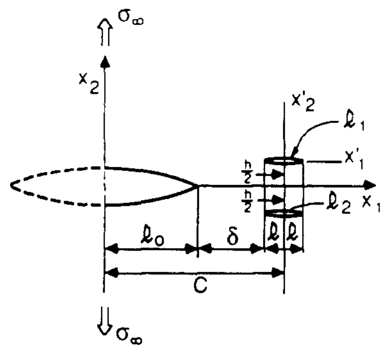

Fig. 7.

$$\mathbf{b}(x') = \frac{4l}{E}\mathbf{n}\cdot\boldsymbol{\sigma}e(x') \tag{45}$$

where $l = l_1 = l_2$ is the length of each microcrack, and E is Young's modulus. The traction vector $\mathbf{n}\cdot\boldsymbol{\sigma}$ is taken at the center of the first microcrack. Thus the components of unknown vector $\mathbf{b}$ are proportional to σ_2 and σ_{22} components of the resulting stress field $\boldsymbol{\sigma}(\mathbf{x})$ at the center of the first microcrack. Using eqn (45) we can rewrite eqn (44) in index form

$$\sigma_{21}\frac{4l}{E}\left[1 + T_{2j}\left\{\int_{l_2} e(x')\phi_{1j}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right] + \sigma_{22}\frac{4l}{E}\left[1 + T_{2j}\left\{\int_{l_2} e(x')\phi_{2j}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right]$$
$$= -K_{\mathrm{I}}^{\mathrm{eff}}\sigma_{0_{2j}}(x) \tag{46}$$

where x is at the center of the first microcrack. Vectorial eqn (46) represents a system of two scalar equations for σ_{21} and σ_{22}

$$\begin{aligned} \alpha_{11}\sigma_{21} + \alpha_{12}\sigma_{22} &= -K_{\mathrm{I}}^{\mathrm{eff}}\sigma_{0_{21}} \\ \alpha_{21}\sigma_{21} + \alpha_{22}\sigma_{22} &= -K_{\mathrm{I}}^{\mathrm{eff}}\sigma_{0_{22}} \end{aligned} \tag{47}$$

where

$$\alpha_{11} = \frac{4l}{E}\left[1 + T_{21}\left\{\int_{l_2} e(x')\phi_{11}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right] \tag{48}$$

$$\alpha_{12} = \frac{4l}{E}\left[1 + T_{21}\left\{\int_{l_2} e(x')\phi_{21}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right] \tag{49}$$

$$\alpha_{21} = \frac{4l}{E}\left[1 + T_{22}\left\{\int_{l_2} e(x')\phi_{12}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right] \tag{50}$$

$$\alpha_{22} = \frac{4l}{E}\left[1 + T_{22}\left\{\int_{l_2} e(x')\phi_{22}(x',\mathbf{x})\,\mathrm{d}x'\right\}\right] \tag{51}$$

and x is the center of the first microcrack.

Solving the system of eqns (47) and substituting the resulting stresses σ_{21} and σ_{22} into the equation for $K_{\mathrm{I}}^{\mathrm{eff}}$ completes the solution. Figure 8 shows $K_{\mathrm{I}}^{\mathrm{eff}}$ as a function of $2l/(h/2)$

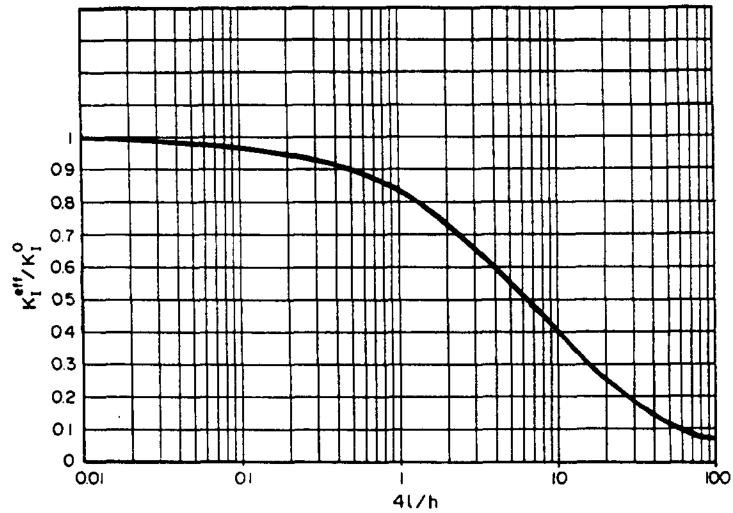

Fig. 8.

and demonstrates the 'shielding' effect for the case $\delta = 0$ when this effect is maximal.

Similar analysis shows that when both microcracks are moved away from the main crack, $K_I^{eff} > K_I^0$ and 'shielding' changes to 'amplification'. In this case, the microcracks have the same effect on K_I^{eff} as one microcrack on the continuation of the main crack line.

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